Exponents

We have seen counting as repeated successors, addition as repeated counting, multiplication as repeated addition, so it is natural to ask what we would get by repeating multiplication.

For example, suppose we repeatedly multiply 2 by itself 3 times:

\[ 2 \times 2 \times 2 \]

The usual notation for this operation is

\[ 2^3 \]

We call 2 the base and 3 the exponent (or power). The operation itself is called exponentiation.

We can do this with other pairs of numbers:

\[ 4^3 = 4 \times 4 \times 4 \]
\[ 3^3 = 3 \times 3 \times 3 \times 3 \times 3 \]
\[ 2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \]

In fact, though we don’t do so here, it can be shown that it is possible to make sense of exponentiation \( a^b \) with any pair of numbers (including fractions and so on) \( a \) and \( b \), as long as the base \( a \) is positive.

The usual notation for the operation of exponentiation is unlike the notations we’ve seen for the operations of addition and multiplication, where we wrote the symbol for the operation (+ or x) between the numbers. Instead, for exponentiation we’ve used a superscript. This can be inconvenient when typing into a computer or in other situations where all symbols must be on the same line, so there is another notation for exponentiation which is commonly used. We write a “hat” symbol, ^, (also know as a “caret”) between the numbers to symbolize the operation of exponentiation:

\[ a \hat{b} = a^b \]

As with addition and multiplication, there are properties of the operation of exponentiation. There is no Law of Commutativity for this operation (since \( 2^3 \neq 3^2 \), that is, 8 ≠ 9) but there is a Law of Identity (since \( a^1 = a \) for any number a).

And there are other properties that are unique to exponentiation. For example, if we multiply

\[ 2^3 \times 2^4 \]

this is the same as

\[ (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) \]

so we multiply 2 by itself \( 3 + 4 \) times.

In other words,

\[ 2^3 \times 2^4 = 2^{3+4} \]

In fact, this is true in general. For a positive base \( a \) and powers \( p \) and \( q \), we have
**Rule 1** \[ a^p \times a^q = a^{p+q} \]

In words, when multiplying two numbers with the same base, we add the exponents.

Similarly, when dividing two numbers with the same base we subtract the exponents:

**Rule 2** \[ \frac{a^p}{a^q} = a^{p-q} \]

Since, for instance, we can cancel 2x2x2 in the numerator and denominator below

\[
\frac{2^5}{2^3} = \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2 \times 2 = 2^{5-3}
\]

Another property of exponentiation involves raising a base to a power to get a certain result, and then raising that result to another power. For example, raise 6 to the power of 2 to get 36:

\[ 6^2 = 36 \]

and then raise 36 to the power of 4:

\[ (6^2)^4 = 36^4 \]

To see the general rule here, note that

\[ (6^2)^4 = 6^2 \times 6^2 \times 6^2 \times 6^2 \]

and by Rule 1

\[ 6^2 \times 6^2 \times 6^2 \times 6^2 = 6^{2+2+2+2} \]

That last exponent is just 2 x 4, so

\[ (6^2)^4 = 6^{2 \times 4} \]

In other words, we multiply the exponents 2 and 4.

The general rule looks like:

**Rule 3** \[ (a^p)^q = a^{pq} \]

If we raise a fraction to a power, we can just the numerator and denominator separately:

\[
\left( \frac{2}{3} \right)^4 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}
\]

and multiplying the fractions we have

\[
\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3} = \frac{2^4}{3^4}
\]

or, in general,

**Rule 4** \[ \left( \frac{a}{b} \right)^p = \frac{a^p}{b^p} \]

As mentioned above, in general a power can be any real number. This poses the question: what is some base to the power of zero?
If we want to understand something like $2^0$
we can’t think of this as multiplying 2 by itself 0 times. Instead we set
$2^0 = 1$
It reason it has to be 1 is to make our other rules work correctly.
If we multiply
$2^0 \times 2^1$
then Rule 1 says that
$2^0 \times 2^1 = 2^{0+1}$
that is,
$2^0 \times 2^1 = 2^1$.
Dividing both sides of this last equation by $2^1$, we cancel the $2^1$’s to get
$2^0 = 1$.

In general,

**Rule 5** \hspace{1cm} a^0 = 1

We also have to make sense of negative exponents. For instance, what is $2^{-1}$?
To make our other rules work correctly we must set
$2^{-1} = \frac{1}{2}$
because when we multiply $2^{-1} \times 2^1$ Rule 1 says
$2^{-1} \times 2^1 = 2^{-1+1}$
that is
$2^{-1} \times 2^1 = 2^0$
then from Rule 5
$2^{-1} \times 2^1 = 1$.
Dividing both sides of this last equation by $2^1$ (which is 2) we have
$2^{-1} = \frac{1}{2}$

In general,

**Rule 6** \hspace{1cm} a^{-1} = \frac{1}{a}

Using Rule 6 together with Rules 3 and 4, we can make sense of any negative exponent:

$a^{-p} = a^{-1 \times p} = (a^{-1})^p = \left(\frac{1}{a}\right)^p = \frac{1}{a^p} = \frac{1}{a^p}$
so we have

**Rule 7** \hspace{1cm} a^{-p} = \frac{1}{a^p}

To summarize:
Rules of Exponents

\[
a^p \times a^q = a^{p+q} \quad \frac{a^p}{a^q} = a^{p-q} \\
(a^p)^q = a^{pq} \quad \left(\frac{a}{b}\right)^p = \frac{a^p}{b^p} \\
a^0 = 1 \quad a^{-1} = \frac{1}{a} \quad a^{-p} = \frac{1}{a^p}
\]

We will use the Rules of Exponents in the next section to do calculations with large numbers. Now we will see how to use the Rules of Exponents to estimate powers of 2.

Powers of 2 come up a lot in Information Technology, as we will see later. Here are the first 10 powers of 2:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( 2^p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
</tr>
<tr>
<td>9</td>
<td>512</td>
</tr>
<tr>
<td>10</td>
<td>1024</td>
</tr>
</tbody>
</table>

Because powers of 2 come up a lot in Information Technology, it’s worth trying to memorize at least those of the above table. But suppose you wanted to know the value of \( 2^{43} \).

We certainly don’t want to memorize the first 43 powers of two! But if we just want a close idea of how much \( 2^{43} \) is then we can use a little trick. Notice from the table above that \( 2^{10} \) is very close to 1000 (which is \( 10^3 \)). In symbols we write \( 2^{10} \approx 10^3 \) (the \( \approx \) means “approximately equals”). Now we can write

\[
2^{43} = 2^{10+10+10+10+3}
\]

and Rule 1 of exponents then says

\[
2^{43} = 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \times 2^3
\]

Approximating each \( 2^{10} \) by \( 10^3 \) we have

\[
2^{43} \approx 10^3 \times 10^3 \times 10^3 \times 10^3 \times 2^3
\]

so

\[
2^{43} \approx 10^{4\times3} \times 2^3
\]

which means

\[
2^{43} \approx 10^{12} \times 8.
\]

Normally we reverse the order of multiplication and write
$2^{43} \approx 8 \times 10^{12}$

The format $8 \times 10^{12}$ is called scientific notation and we will study it in the next section.

All of this illustrates:

<table>
<thead>
<tr>
<th><strong>Method for approximating powers of two</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>To estimate $2^n$ for some natural number $n$, write</td>
</tr>
<tr>
<td>$N = m \times 10 + r$</td>
</tr>
<tr>
<td>Where $m$ is the number of times that $10$ goes into $N$ and $r$ is the remainder.</td>
</tr>
<tr>
<td>Then</td>
</tr>
<tr>
<td>$2^n \approx 2^r \times 10^mx^3$</td>
</tr>
</tbody>
</table>

If you memorize the first ten powers of $2$ you can, using this method, quickly estimate any power of $2$ in your head.