Business Mathematics I

SOLUTIONS TO STUDY GUIDE FOR MIDTERM 2

The following questions explore parts of the material that will be covered on Test 2.

Problems 1, 2, and 3 refer to a sum of $20,000 which is deposited in a savings account that pays an annual rate of 4.9%.

1. What is the effective annual yield on the account, if interest is compounded continuously? Give your answer as a percentage, rounded to 2 decimal places, e.g., x.xx%.

**Solution.** \( y = e^r - 1 = e^{0.049} - 1 \approx 0.0502 = 5.02\% \)

2. What is the future value of the account after 5 years, if interest is compounded continuously?

**Solution.** \( F = P \cdot e^{rt} = 20,000 \cdot e^{0.049 \cdot 5} \approx 25,552.43 \)

3. What is the future value of the account after 5 years, if interest is compounded quarterly?

**Solution.** \( F = P \cdot \left(1 + \frac{r}{n}\right)^{nt} = 20,000 \cdot \left(1 + \frac{0.049}{4}\right)^{4 \cdot 5} = 25,514.42 \)

4. If interest is compounded continuously at 6.5%, what is the present value of a future payment of $50,000 that is to be made 8 years from now?

**Solution.** \( P = \frac{F}{e^{-rt}} = 50,000 \cdot e^{-0.065 \cdot 8} \approx 29,726.03 \)

5. Fill in the **Frequency**, **Relative Frequency**, and **Percentage** columns, as they would be done by the *Histogram* and other functions in *Excel*. Do this manually, not with the help of *Excel*.

| Data | Bins | Frequency | Relative Frequency | Percentage |
|------|------|-----------|--------------------|------------|------------|
|      |      |           |                    |            |            |
Problems 6 and 7 refer to the following plot of ratios for 600 weekly closes on a stock.

### Ratios of Closing Prices

<table>
<thead>
<tr>
<th>ratio</th>
<th>percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.83</td>
<td>0%</td>
</tr>
<tr>
<td>0.86</td>
<td>5%</td>
</tr>
<tr>
<td>0.89</td>
<td>10%</td>
</tr>
<tr>
<td>0.92</td>
<td>15%</td>
</tr>
<tr>
<td>0.95</td>
<td>20%</td>
</tr>
<tr>
<td>0.98</td>
<td>25%</td>
</tr>
<tr>
<td>1.01</td>
<td>30%</td>
</tr>
<tr>
<td>1.04</td>
<td>35%</td>
</tr>
<tr>
<td>1.07</td>
<td>30%</td>
</tr>
<tr>
<td>1.10</td>
<td>25%</td>
</tr>
<tr>
<td>1.13</td>
<td>20%</td>
</tr>
<tr>
<td>1.16</td>
<td>15%</td>
</tr>
<tr>
<td>1.19</td>
<td>10%</td>
</tr>
</tbody>
</table>

6. Approximately how many ratios were between 0.995 and 1.025?

**Solution.** Approximately $0.31 \times 600 = 186$ ratios.

7. Assuming that these ratios are a random sample for the random variable $R$, which gives the weekly ratio of closes, estimate $P(R \geq 1.025)$.

**Solution.**

$$P(R \geq 1.025) \equiv \frac{\text{number of ratios} \geq 1.025}{600}$$

$$\equiv \frac{0.22 + 0.08 + 0.03 + 0.01}{600} = 0.34$$

8. Let $X$ be the random variable giving the value of the S and P 500 Index on July 1, 2001. Is $X$ finite or continuous? Is $f_X$ a p.m.f. or a p.d.f.?

**Solution.** $X$ is a continuous random variable, with a p.d.f.
9. Let \( Y \) be the random variable giving the number of months in the year 2001 when the S and P 500 Index has a monthly high above 1,600. Is \( Y \) finite or continuous? Is \( f_Y \) a p.m.f. or a p.d.f.?

**Solution.** \( Y \) is a finite random variable, with a p.m.f.

10. For each graph, select the statement that is most likely to be correct.

   a. The function is a **probability mass function** for a **finite** random variable.
   b. The function is a **probability density function** for a **continuous** random variable.
   c. The function is a **cumulative distribution function** for a **finite** random variable.
   d. The function is a **cumulative distribution function** for a **continuous** random variable.
   e. None of the above.

**Hint:** look carefully at the numbers on the axes.

(i)

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 0.2 & 0.4 & 0.6 & \hline
1 & \hline
2 & \hline
3 & \hline
\end{array}
\]

\[\text{x}\]  a.  \[\_\_\_\_\]  b.  \[\_\_\_\_\]  c.  \[\_\_\_\_\]  d.  \[\_\_\_\_\]  e.  \[\_\_\_\_\]

(ii)

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & \hline
0.2 & \hline
0.4 & \hline
0.6 & \hline
0.8 & \hline
1.0 & \hline
1.2 & \hline
\end{array}
\]

\[\_\_\_\_\]  a.  \[\_\_\_\_\]  b.  \[\_\_\_\_\]  c.  \[\_\_\_\_\]  d.  \[\_\_\_\_\]  e.  \[\_\_\_\_\]

(iii)

\[
\begin{array}{cccccc}
-0.5 & 0 & 0.5 & 1 & 1.5 & \hline
0 & \hline
0.2 & \hline
0.4 & \hline
0.6 & \hline
0.8 & \hline
1.0 & \hline
1.2 & \hline
\end{array}
\]

\[\_\_\_\_\]  a.  \[\_\_\_\_\]  b.  \[\_\_\_\_\]  c.  \[\_\_\_\_\]  d.  \[\_\_\_\_\]  e.  \[\_\_\_\_\]
10. A random variable $W$ has the following values and probabilities.

<table>
<thead>
<tr>
<th>$w$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(W = w)$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>

(i) The c.d.f., $F_W$ has $F_W(8) = \_0.9\_.$

(ii) The p.m.f., $f_W$ has $f_W(8) = \_0.3\_.$

(iii) The c.d.f., $F_W$ has $F_W(10) = \_0.9\_.$

(iv) The p.m.f., $f_W$ has $f_W(10) = \_0\_.$

11. A courier van that runs between two of your company's facilities encounters 5 traffic signals along its route. These operate independently, with each signal having a probability of 0.55 that the van has to stop. Let $X$ be the random variable giving the number of signals at which the van must stop on a randomly selected trip.

(i) What kind of random variable is $X$?

(ii) Use Excel to generate a table containing all possible values of $X$ and the corresponding values of $f_X$ and $F_X$, rounded to 4 decimal places.

(iii) On average, at how many of the signals can the courier van expect to stop? Use the most efficient method for computing your answer.

**Solution.** (i) $X$ is a finite random variable, having a binomial distribution, with parameters $n = 5$ and $p = 0.55$. 
(ii) **BINOMDIST** was used to create the following table.

<table>
<thead>
<tr>
<th>Binomial X</th>
<th>$f_X(x)$</th>
<th>$F_X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0185</td>
<td>0.0185</td>
</tr>
<tr>
<td>1</td>
<td>0.1128</td>
<td>0.1312</td>
</tr>
<tr>
<td>2</td>
<td>0.2757</td>
<td>0.4069</td>
</tr>
<tr>
<td>3</td>
<td>0.3369</td>
<td>0.7438</td>
</tr>
<tr>
<td>4</td>
<td>0.2059</td>
<td>0.9497</td>
</tr>
<tr>
<td>5</td>
<td>0.0503</td>
<td>1.0000</td>
</tr>
<tr>
<td>Sum</td>
<td>1.0000</td>
<td></td>
</tr>
</tbody>
</table>

(iii) On average, the courier van can expect to stop at $E(X)$ signals. $E(X) = np = 5 \cdot 0.55 = 2.75$ stops.

12. $T$ is a uniform continuous random variable on the interval [0, 5].

(i) The **p.d.f.**, $f_T$, has $f_T(3) = \frac{1}{0.2}$. $P(T = 3) = \frac{0}{0}$.

13. Let $X$ be an exponential random variable, whose **p.d.f.** has the form $f_X(x) = 0.2 e^{-0.2x}$, for $x \geq 0$.

(i) Find a formula for the **c.d.f.** of $X$.

(ii) Use your result from Part i to compute $P(3.7 \leq X \leq 6.7)$, rounded to 4 decimal places.

(iii) $\mu_X =$ ?

**Solution.** (i) For $x \geq 0$, the **p.d.f.** has the form $f_X(x) = \frac{1}{\alpha} e^{-x/\alpha}$ with $\alpha = 5$. Hence, $X$ is exponential with parameter $\alpha = 5$, and the following **c.d.f.**

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-e^{-x/5} & \text{if } 0 \leq x \end{cases}$$

(ii) $P(3.7 \leq X \leq 6.7) = F_X(6.7) - F_X(3.7) = (1 - e^{-6.7/5}) - (1 - e^{-3.7/5}) = e^{-3.7/5} - e^{-6.7/5} \approx 0.477113916 - 0.261845669 = 0.215268247 \approx 0.2153$, rounded to 4 decimal places.

(iii) $\mu_X = \alpha = 5$.

14. For each of the following functions, tell were it is located in **Excel**, and describe its use.

(i) **VLOOKUP**

(ii) **IF**

(iii) **RAND**

(iv) **HISTOGRAM**

(v) **RANDBETWEEN**

(vi) **SORT**
**Solution.** (i) VLOOKUP is found in the Lookup and reference functions. It is used to find the number in a given column, based upon the value in another column.

(ii) IF is found in Logical functions. It is used to choose between two expressions, depending upon the truth of a given statement.

(iii) RAND is found in the Math & trig functions. It is used to generate an observation of a continuous random variable which is uniform on [0, 1].

(iv) HISTOGRAM is found under Tools/Data Analysis. It is used to count the number of data points which lie within specified ranges.

(v) RANDBETWEEN is found in the Math & trig functions. It is used to generate random integers between given limits.

(vi) SORT is found under Data. It is used to sort data points by columns or rows.

**Problems 15, 16, and 17 refer to a 10 week European call option on ABC stock with a strike price of $90.**

15. A few of the historical adjusted weekly closing prices are shown below.

<table>
<thead>
<tr>
<th>Week of</th>
<th>Adjusted Close</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nov. 15, 1999</td>
<td>$86.89</td>
<td>1.0109</td>
</tr>
<tr>
<td>Nov. 8, 1999</td>
<td>$85.95</td>
<td>1.0198</td>
</tr>
<tr>
<td>Nov. 1, 1999</td>
<td>$84.28</td>
<td>0.9800</td>
</tr>
<tr>
<td>Oct. 25, 1999</td>
<td>$86.00</td>
<td>1.0126</td>
</tr>
<tr>
<td>Oct. 18, 1999</td>
<td>$84.93</td>
<td></td>
</tr>
</tbody>
</table>

Fill in the column with ratios of weekly closes and compute the mean ratio for the given period.

**Solution.** The mean ratio is \((1.0109 + 1.0198 + 0.9800 + 1.0126)/4 = 1.0058\)

16. Suppose that the only possible closing prices of ABC, at the end of the option, are $84, $91, and $93. If the three closing values are all equally likely, what is the expected value of C, the closing value? What does \(E(C)\) tell you about the future value of the option?

**Solution.** \(E(C) = \$84(1/3) + \$91(1/3) + \$93(1/3) = \$89.33\). This number does not tell us anything about the future value of the option.

17. Under the conditions in Problem 14, what is the expected future value of the option?

**Solution.** Let \(F\) give the future value of the option. \(E(F) = \$0(1/3) + \$1(1/3) + \$3(1/3) = \$1.33\).